Fluctuational phase-flip transitions in parametrically pumped oscillators

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We analyze the rates of noise-induced transitions between period-two attractors. The model investigated is an underdamped oscillator parametrically driven by a field at nearly twice the oscillator eigenfrequency. The activation energy of the transitions is analyzed as a function of the frequency detuning and field amplitude scaled by the damping and nonlinearity parameters of the oscillator. The parameter ranges where the system is bi- and tristable are investigated. Explicit results are obtained in the limit of small damping (or strong driving), and near bifurcation points.

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I. INTRODUCTION

Nonlinear systems driven by a sufficiently strong periodic field often display period doubling [1]. The two emerging stable periodic states are the same: the only difference between them is that they are shifted in time by the period of the force $2\pi/\omega_F$. This feature is a consequence of the symmetry with respect to translation in time by $2\pi/\omega_F$, and it has attracted much attention to such systems as elements of digital computers. In the presence of noise, there occur fluctuational transitions between the period-two attractors, which correspond to phase slip of the system by π. Such transitions have been observed recently for electrons oscillating in a Penning trap [2]. They have been also investigated in an analog electronic circuit [3], in the context of stochastic resonance [4].

Motivated by these observations, in the present paper we develop a theory of escape rates from period-two attractors. The analysis is done for the simplest generic model that displays period doubling: an underdamped oscillator parametrically driven by a force at nearly twice the oscillator eigenfrequency ω_0 [5]. This model applies, in particular, to axial vibrations of an electron in a Penning trap [2]. Much work on a parametrically excited oscillator has been done in the context of squeezed states of light, cf. [6]. We analyze escape due to classical fluctuations, which were substantial for the systems investigated in [2,3].

Parametrically excited oscillator is an example of a system away from thermal equilibrium. Such systems usually lack detailed balance [7], they are not characterized by free energy, and escape rates depend on the system dynamics and the noise that gives rise to fluctuations in the system. In the important and quite general case where the noise is Gaussian, there has been developed a technique which reduces the problem of calculating escape rates to a variational problem [8]. The solution of this problem describes the optimal path along which the fluctuating system is most likely to move when it escapes. Such path is often called the most probable escape path (MPEP) [9]. The value of the variational functional gives the exponent in the expression for the escape rate.

An advantageous feature of the problem of period doubling in an underdamped oscillator is that the period doubling occurs for comparatively small amplitudes F of the driving force, where the nonlinearity of the oscillator is still small: the anharmonic part of the potential energy is much less than the quadratic term $\omega_0^2 q^2/2$. In this case, the quantities of interest are the amplitude and phase of the vibrations at the frequency $\omega_F/2 \approx \omega_0$. They vary only a little over the time $\sim \omega_F^{-1}$. The corresponding dynamics is affected by Fourier components of the noise within a narrow band centered at $\omega_F/2$. Essentially, this means that, in the analysis of the dynamics of slow variables, the noise may be assumed white. A similar situation arises [10] in the problem of transitions between the stable states of forced vibrations of a resonantly driven underdamped oscillator.

Below in Sec. II we provide the results on the the phase portrait of the oscillator in the rotating frame, derive the properties of noise for slow variables, and, for low noise intensities, formulate the variational problem for the activation energy of escape. In Sec. III we provide explicit expressions for the escape rates in the following parameter ranges: vicinities of the bifurcation points where period two attractors emerge and where there emerge two unstable period two states, and the range of comparatively strong driving where the motion in *slow* variables is underdamped. In Sec. IV numerical results for the general case are discussed and compared with the results for the limiting cases. Sec. V contains concluding remarks.

II. ASYMPTOTIC EXPRESSION FOR THE ESCAPE RATES

A. Phase portrait in slow variables

To set the scene, we will first discuss the phase portrait of a parametrically driven underdamped oscillator.

A simple phenomenological equation of motion is of the form

$$\frac{d^2q}{dt^2}+2\Gamma\frac{dq}{dt}+\omega_0^2q+\gamma q^3+qF\cos\omega_F t=\xi(t) \qquad (1)$$

Here, Γ is the friction coefficient, γ is the nonlinearity parameter, F is the amplitude of the regular force, and $\xi(t)$ is a zero-mean noise, $\langle\xi(t)\rangle=0$. The effect of cubic nonlinearity in the Hamiltonian of the oscillator can be taken into account by renormalizing the parameter γ (cf. [5]). The results can be also easily generalized to the case where, for special reasons [2], γ is numerically small, and it is necessary to allow for the term $\propto q^5$ in the equation of motion.

We consider resonant driving, so that the oscillator eigenfrequency ω_0 is close to $\omega_F/2$,

$$\Gamma, |2\omega_0-\omega_F| \ll \omega_0. \qquad (2)$$

In this case it is convenient (cf. [5]) to analyze the oscillator motion in the rotating frame. Respectively, we change to slow dimensionless time $\tau=\Gamma t$ and slow dimensionless variables q_1, q_2:

$$q(t)=\left(\frac{4\omega_F\Gamma}{3|\gamma|}\right)^{1/2}\left[q_1\cos\frac{\omega_F t}{2}-q_2\sin\frac{\omega_F t}{2}\right], \qquad (3)$$
$$\frac{dq}{dt}=-\left(\frac{\omega_F^3\Gamma}{3|\gamma|}\right)^{1/2}\left[q_1\sin\frac{\omega_F t}{2}+q_2\cos\frac{\omega_F t}{2}\right].$$

Following the standard procedure of the method of averaging (cf. [1]) and neglecting fast oscillating terms which depend on the oscillator amplitude and contain a factor $\exp(\pm in\omega_F t/2)$ with $n\neq 0$, one obtains the equations of motion for q_1, q_2 in the form

$$\dot q_1\equiv\frac{dq_1}{d\tau}=-q_1+\frac{\partial g}{\partial q_2}+\xi_1(\tau/\Gamma),\ \tau\equiv\Gamma t, \qquad (4)$$
$$\dot q_2\equiv\frac{dq_2}{d\tau}=-q_2-\frac{\partial g}{\partial q_1}+\xi_2(\tau/\Gamma)$$

where $\xi_{1,2}(\tau/\Gamma)$ are the random forces, and

$$g(q_1,q_2)=\frac{1}{2}(q_1^2+q_2^2)\left[\Omega-\frac{1}{2}(q_1^2+q_2^2)\operatorname{sgn}\gamma\right] \qquad (5)$$
$$+\frac{1}{2}\zeta(q_2^2-q_1^2).$$

In what follows we assume that $\gamma>0$; the case $\gamma<0$ can be described by replacing $\Omega\to-\Omega$, $q_i\to -q_i$ $(i=1,2)$, and by changing the phase of the field in Eq. (1) by π.

Except for the random force, the motion of the oscillator as described by Eqs. (4) is characterized by two dimensionless parameters: the scaled frequency detuning Ω and the scaled field ζ,

$$\Omega=[(\omega_F/2)-\omega_0]/\Gamma, \quad \zeta=F/2\omega_F\Gamma. \qquad (6)$$

For $\zeta<1$ or for $\Omega<-(\zeta^2-1)^{1/2}$ the oscillator (4) in the absence of noise has only one stable state, $q_1=q_2=0$: parametric oscillations are not excited. The value $\zeta=1$ gives the threshold field amplitude $F_{th}=2\omega_F$ which is necessary for exciting period-two vibrations. The phase portrait of the oscillator in variables q_1, q_2 in the range where the oscillations are excited is shown in Fig. 1.

For $\zeta>1$, with the increasing Ω there first occurs a pitchfork bifurcation for $\Omega=-(\zeta^2-1)^{1/2}$. Here, the stable state $q_1=q_2=0$ becomes unstable and there emerge two stable states $\mathbf{q}_{st}^{(1,2)}$ which are symmetric with respect to $q_1=q_2=0$, see Fig. 1a. These states correspond to stable period two vibrations shifted in phase by π. The vibrational amplitude $a=|\mathbf{q}_{st}|$ increases monotonically with the increasing Ω (see inset in Fig. 1). The results of the asymptotic analysis based on Eqs. (4) apply for not too large amplitudes a,

$$\Gamma a^2, |2\omega_F-\omega_0|a^2 \ll \omega_F. \qquad (7)$$

As Ω goes through the value $(\zeta^2-1)^{1/2}$ there occurs the second pitchfork bifurcation of the state $q_1=q_2=0$: this state becomes stable again, and there emerge two unstable states $\mathbf{q}_u^{(1,2)}$, which are also symmetric with respect to the origin and correspond to unstable period two vibrations of the oscillator. As it is seen from Fig. 1b, the phase plane of the oscillator is such that, for $\Omega>(\zeta^2-1)^{1/2}$ the domains of attraction to the stable states $\mathbf{q}_{st}^{(1)}, \mathbf{q}_{st}^{(2)}$ are separated by the domain of attraction to the stable state $\mathbf{q}=0$.

B. Noise in the equations for slow variables

In many cases of physical interest, the noise $\xi(t)$ driving the oscillator is Gaussian. This noise may originate from the coupling to a thermal bath (which also gives rise to the friction force in Eq. (1)), or it may be due to an external nonthermal source. A zero-mean Gaussian noise is characterized by its power spectrum

$$\Phi_\omega[\xi(t+t'),\xi(t)]=\phi(\omega)=\int_{-\infty}^{\infty}dt'\,e^{i\omega t'}\langle\xi(t+t')\xi(t)\rangle. \qquad (8)$$

For a stationary noise the power spectrum (8) is independent of time. In what follows we assume that the function $\phi(\omega)$ is *smooth* near the oscillator eigenfrequency ω_0.

The noises $\xi_{1,2}(t)$ in Eq. (4) are nonstationary, generally speaking. One obtains from Eqs. (1), (3), (4) the following expressions for their power spectra:

$$\Phi_\omega[\xi_i(t+t'),\xi_i(t)]=\frac{3|\gamma|}{8\omega_F^3\Gamma}\phi\left(\omega-\frac{1}{2}\omega_F\right)$$
$$\times\left[1-(-1)^i\cos\omega_F t\right]\ (i=1,2), \qquad (9)$$
$$\Phi_\omega[\xi_1(t+t'),\xi_2(t)]=-\frac{3|\gamma|}{8\omega_F^3\Gamma}\phi\left(\omega-\frac{1}{2}\omega_F\right)\sin\omega_F t.$$

It follows from Eq. (9) that the spectra of the diagonal correlators $\langle \xi_i(t)\xi_i(t')\rangle$ have both time-independent and fast-oscillating in time components, whereas the power spectrum of the cross-correlator of ξ_1, ξ_2 is fast oscillating in time. Therefore, in the analysis of the effect of the noise on the slowly varying functions $q_{1,2}$, in the spirit of the averaging method, one can assume that the noise components $\xi_1(t), \xi_2(t)$ are asymptotically independent of each other; one can also leave out the terms $\propto \cos\omega_F t$ in the power spectra of the diagonal correlators (cf. [10] where the analysis was done for a microscopic model of noise resulting from coupling to a bath; rigorous mathematical results on the method of averaging in noise-driven systems are discussed in [11]).

The dynamics of the slow variables $q_{1,2}$ is characterized by the time scales $\sim 1/\Gamma, 1/|2\omega_F - \omega_0|$. If the power spectrum $\phi(\omega)$ varies only slightly in the whole range $|\omega - \omega_F/2| \lesssim \Gamma, |2\omega_F - \omega_0|$ (and this range does not correspond to a deep minimum of $\phi(\omega)$), then one can assume that the random functions $\xi_1(t)$, $\xi_2(t)$ are asymptotically independent zero-mean Gaussian white noises,

$$\langle \xi_i(\tau/\Gamma)\xi_i(\tau'/\Gamma)\rangle \approx D\tilde{\delta}(\tau - \tau'),\ i = 1, 2 \qquad (10)$$
$$D = (3|\gamma|/8\omega_F^3\Gamma^2)\phi(\omega_F/2)$$

(the function $\tilde{\delta}(x)$ is of the type of a δ-function: it is large in the domain $|x| \ll 1$, and its integral is equal to 1).

The quantity D gives the characteristic intensity of the noise in the equations of motion for slow variables q_1, q_2 (4). In particular, if the oscillator is coupled to a thermal bath with the correlation time much smaller than $1/\omega_0$, so that it performs a "truly" Brownian motion, and the random force $\xi(t)$ in Eq. (1) is δ-correlated, with intensity $4\Gamma kT$, we have $D = 3|\gamma|kT/2\omega_F^3\Gamma$. We note, however, that the dynamics of slow variables can be described as Brownian motion even where this description does not apply to the motion of the initial oscillator, i.e. where the correlation time of the thermal bath is $\gtrsim 1/\omega_0$. In this latter case, the friction force in Eq. (1) is also retarded (cf. [10]).

C. Activation energy of escape

If the dimensionless noise intensity D is small, for most of the time the oscillator fluctuates in a small vicinity of one or the other stable state $\mathbf{q}_{st}^{(n)}$. Only occasionally there occurs a large fluctuation which results in a transition to another stable state. The probability W_n of such fluctuation is exponentially small, and its dependence on the noise intensity is given by the activation law, $W_n \propto \exp(-S_n/D)$ (see [8] for a review). In fact, to logarithmic accuracy W_n is determined by the probability density of the least improbable realization of the force $\boldsymbol{\xi}(\tau/\Gamma)$ which results in the corresponding transition. Therefore the quantity S_n is given by the solution of a variational problem. This problem is of the form (cf. [10,11]):

$$W_n = C\exp(-S_n/D),\ S_n = \min S[\mathbf{q}(\tau)], \qquad (11)$$
$$S[\mathbf{q}(t)] = \int_{-\infty}^{\infty} d\tau\, L(\dot{\mathbf{q}}, \mathbf{q}),\ L(\dot{\mathbf{q}}, \mathbf{q}) = \frac{1}{2}\left[\dot{\mathbf{q}} - \mathbf{K}(\mathbf{q})\right]^2,$$
$$\mathbf{q}(t) \to \mathbf{q}_{st}^{(n)} \text{ for } t \to -\infty, \quad \mathbf{q}(t) \to \mathbf{q}_u \text{ for } t \to \infty,$$

where the components of the vector $\mathbf{K}$ are given by the rhs parts of the equations of motion (4),

$$K_1(\mathbf{q}) = -q_1 + \frac{\partial g}{\partial q_2},\ K_2(\mathbf{q}) = -q_2 - \frac{\partial g}{\partial q_1} \qquad (12)$$

(in what follows we set $n = 1, 2$ for the stable states of period two vibrations, and $n = 0$ for the stationary state $\mathbf{q} = 0$ where it is stable).

The solution of the variational problem (11) $\mathbf{q}(t)$ describes the optimal, or most probable escape path (MPEP) from the nth stable state. This path is instanton-like (see [12]): it starts at the stable state $\mathbf{q}_{st}^{(n)}$ for $t \to -\infty$, and for $t \to \infty$ it approaches the unstable state $\mathbf{q}_u$ on the boundary of the domain of attraction to $\mathbf{q}_{st}^{(n)}$ (having reached the boundary, the system makes a transition to another stable state with a probability $\sim 1/2$). It follows from Fig. 1 that, for $-(\zeta^2 - 1)^{1/2} < \Omega < (\zeta^2 - 1)^{1/2}$, where the only stable states are period two attractors, escape from one of them means a transition to the other. For $\Omega > (\zeta^2 - 1)^{1/2}$ escape from one of period two attractors means a transition to the state where period two vibrations are not excited. From this state, the system makes fluctuational transitions into one or the other state of period two vibrations.

The most probable realization of the force is related to the optimal fluctuational path via Eq. (4), $\boldsymbol{\xi}(\tau/\Gamma) = \dot{\mathbf{q}} - \mathbf{K}$. Optimal fluctuational paths are physically real, they have been observed in the experiment [13].

III. ACTIVATION ENERGY OF ESCAPE IN THE VICINITIES OF THE BIFURCATION POINTS

The activation energies S_n, as defined by (11), depend on two dimensionless parameters of the driven oscillator: the scaled field strength ζ and the scaled frequency detuning Ω (6). In the general case, S_n may be calculated numerically (see Sec. VI). Explicit expressions for S_n can be obtained in several limiting cases.

In the range of comparatively small nonlinearity (7), the oscillator may experience two bifurcations with the varying field frequency, as seen from Fig. 1. In the vicinity of a bifurcation point, one of the motions of the system near the emerging stable state(s) becomes slow: there arises a "soft mode" [14]. Correspondingly, fluctuations near a bifurcation point have universal features [15].

Near a bifurcation point one can either solve the variational problem (11) explicitly or reduce the system of equations of motion (4) to one first order equation. From this equation one can find not only the exponent, but also the prefactor in the expression for the escape rate.

The equation for a slow variable Q can be derived from (4) by appropriately rotating the coordinates:

$$Q = q_1 \cos\beta + q_2 \sin\beta,\ P = -q_1 \sin\beta + q_2 \cos\beta \qquad (13)$$
$$\tan 2\beta = -\Omega_B^{-1},\ \Omega_B = \mp(\zeta^2 - 1)^{1/2}$$

(Ω_B are the bifurcation values of the dimensionless frequency detuning, see Fig. 1c). For $|\Omega - \Omega_B| \to 0$ the dimensionless relaxation time of the variable Q goes to infinity, whereas that of P is 1/2, and therefore $P(\tau)$ follows the slow variable $Q(\tau)$ adiabatically [15]. The fluctuations in P can be neglected compared to the fluctuations in Q. Using the adiabatic solution for P (i.e., neglecting $\dot{P}$ and the noise term in the equation for $\dot{P}$) we obtain the following equation for Q:

$$\dot{Q} = -dU/dQ + \Xi(\tau),\ \langle \Xi(\tau)\Xi(\tau')\rangle = D\tilde{\delta}(\tau - \tau') \qquad (14)$$
$$U(Q) = \Omega_B \left[\frac{1}{2}(\Omega - \Omega_B)Q^2 - \frac{1}{4}\zeta^2 Q^4\right],\ \Omega_B|\Omega - \Omega_B| \ll 1.$$

It is seen from Eq. (14) that, near the bifurcation point $\Omega_B = -(\zeta^2 - 1)^{1/2}$, the system has one stable state for $\Omega < \Omega_B$ and two symmetrical stable states for $\Omega > \Omega_B$ (cf. Fig. 1). The escape rates from these symmetrical states are the same and are given by the Kramers expression [16]

$$W_n = (\sqrt{2}/\pi)|\Omega_B|(\Omega - \Omega_B)\exp(-S_n/D), \qquad (15)$$
$$S_n = 2\left[U(Q^{(n)}) - U(Q^{(u)})\right] = |\Omega_B|(\Omega - \Omega_B)^2/2\zeta^2$$

($Q^{(n)}$ and $Q^{(u)}$ are the values of the coordinate Q in the stable and unstable states).

The rate (15) is the rate at which there occur phase slip transitions between the period two stable states for Ω close to $\Omega_B = -(\zeta^2 - 1)^{1/2}$. Eq. (15) applies for $\exp(-S_n/D) \ll 1$. The activation energy S_n is quadratic in the distance $|\Omega - \Omega_B|$ to the bifurcation point. For a given $|\Omega - \Omega_B|$ the dependence of S_n on the dimensionless field ζ is given by the factor $(\zeta^2 - 1)^{1/2}/\zeta^2$.

For $0 < \Omega - (\zeta^2 - 1)^{1/2} \ll 1$, Eq. (15) describes the rate of transitions from the stable state $\mathbf{q}_{st} = 0$ to each of the period two stable states. The transitions occur via the appropriate unstable period two state (the one on the boundary between the domain of attractions to the stable period two state and the stable state $\mathbf{q} = 0$).

IV. ACTIVATION ENERGIES OF ESCAPE IN THE SMALL-DAMPING LIMIT

A. Motion in the absence of dissipation

Of special interest is the case where the scaled field amplitude ζ is large enough so that the dissipation terms $-q_1, -q_2$ in the rhs of the equations for slow variables (4) are comparatively small. In the neglect of these terms and the random force, Eqs. (4) describe conservative motion of a particle with the coordinate q_1 and momentum q_2, and with the Hamiltonian function $g(q_1, q_2)$ (5). This particle moves along closed trajectories shown in Fig. 2 for the auxiliary variables X, Y:

$$X = q_1/\zeta^{1/2},\ Y = q_2/\zeta^{1/2},$$
$$\frac{dX}{d\tilde{\tau}} = \frac{\partial G}{\partial Y},\ \frac{dY}{d\tilde{\tau}} = -\frac{\partial G}{\partial X},\ \tilde{\tau} = \zeta\tau. \qquad (16)$$
$$G(X,Y) = \zeta^{-2} g\left(\zeta^{1/2}X, \zeta^{1/2}Y\right) = \frac{1}{2}(\mu - 1)X^2$$
$$+\frac{1}{2}(\mu + 1)Y^2 - \frac{1}{4}(X^2 + Y^2)^2,\ \mu = \frac{\Omega}{\zeta}.$$

The conservative motion (16) depends only on one parameter, $\mu = \Omega/\zeta$, which characterizes the interrelation between the frequency detuning and the field strength. The shape of the effective energy $G(X, Y)$ for two different values of μ is shown in Fig. 3. The trajectories in Fig. 2 are just the cross-sections of the surface $G(X, Y)$ by planes $G =$const. The extrema of the surface $G(X, Y)$ are the fixed points of the system.

For $\mu < -1$ the surface $G(X, Y)$ has one extremum placed at $X = Y = 0$, which corresponds, with dissipation taken into account, to the stable state with no period two vibrations excited. For $-1 < \mu < 1$ the function $G(X, Y)$ has two maxima at $X = 0$, $Y = \pm(\mu+1)^{1/2}$ (they correspond to two period two attractors) and a saddle point at $X = Y = 0$. For $\mu > 1$, in addition to the above maxima the function $G(X, Y)$ has a minimum at $X = Y = 0$ (the maxima and the minimum correspond to the stable states of the oscillator), and two saddle points at $X = \pm(\mu - 1)^{1/2}, Y = 0$. The extreme values of G are given by the expressions

$$G_{1,2} = \frac{1}{4}(\mu + 1)^2,\ G_u = 0 \text{ for } \mu < 1; \qquad (17)$$
$$G_0 = 0,\ G_u = \frac{1}{4}(\mu - 1)^2 \text{ for } \mu > 1.$$

We note that for $\mu > 1$ the trajectories surrounding the states at $X = 0$, $Y = \pm(\mu+1)^{1/2}$ in Fig. 2 become horseshoe like for large enough $G_1 - G$, and also that for $0 \leq G \leq G_u$ there are coexisting “internal” and “external” trajectories.

B. Escape rates

The effect of small dissipation in Eqs. (4) is to transform the closed trajectories in Fig. 2 into small-step spirals which wind down to the corresponding stable states

(cf. Fig. 1). The motion can be described in a standard way in terms of slow drift over the energy g towards the stable state (cf. [1]).

The random force which drives the system away from the stable state should "beat" this drift. It would be expected that the optimal fluctuational path corresponds to energy diffusion *away* from the stable state. A solution of the variational problem (11) for small dissipation was obtained in [10] for a different form of the function $g(q_1, q_2)$. Alternatively, one can use the approach first suggested by Kramers [16] in the analysis of escape of underdamped thermal equilibrium systems. This approach is equivalent to deriving from Eqs. (4) an equation for $\dot{g}$ and averaging the dissipative term and the noise intensity in this equation over the period of vibrations with a given g in the absence of dissipation and noise. The dissipative term (the dissipation rate of g) is then given by the expression $-\overline{[q_1(\partial g/\partial q_1) + q_2(\partial g/\partial q_2)]}$, with an accuracy to the correction $\propto D$ (here, overline means averaging over the vibration period). The diffusion coefficient for g is given by $D\overline{[\partial g/\partial q_1)^2 + (\partial g/\partial q_2)^2]}$. The resulting first order equation for g can be solved to give the following expression for the activation energy:

$$S_n = 2\zeta \int_{G_n}^{G_u} dG \, \frac{M(G)}{N(G)}, \; M(G) = \iint_{A(G)} dX dY,$$
$$N(G) = \frac{1}{2} \iint_{A(G)} dX dY \, \nabla^2 G(X,Y). \tag{18}$$

Using the explicit form of $G(X,Y)$ one obtains

$$N(G) = \iint_{A(G)} dX dY \, \left[\mu - 2(X^2+Y^2)\right], \tag{19}$$

The values of G_n (n =1,2) and G_u are the extreme values of $G(X,Y)$ as given by Eq. (17). The double integrals in (18) are taken over the areas $A(G)$ limited by the trajectories $G(X,Y) = G$ in Fig. 2 which surround the nth center (the expressions for $M(G), N(G)$ were obtained from the expressions for the drift and diffusion coefficients for g using the Stocks theorem, with account taken of Eqs.(16), as it was done in [10]).

The expressions for $M(G), N(G)$ (18) can be further simplified by changing to polar coordinates $X = R\cos\varphi$, $Y = R\sin\varphi$. Solving Eq. (16) for R^2 in terms of G, φ and integrating over R^2 one then obtains that, in the problem of the escape from the period two attractors ($n = 1, 2$ in (18), i.e., $G \geq G_u$, cf. Fig. 3),

$$M(G) = \int d\varphi \, f(G,\varphi), \tag{20}$$
$$N(G) = -\int d\varphi \, (\mu - 2\cos 2\varphi) f(G,\varphi),$$
$$f(G,\varphi) = \left[(\mu - \cos 2\varphi)^2 - 4G\right]^{1/2} \; (G \geq G_u).$$

The limits of the integrals over φ are determined from the condition that $f(G,\varphi)$ be real and $\mu - \cos 2\varphi > 0$.

In the problem of escape from the stable state at $\mathbf{q} = 0$ for $\mu > 1$ ($G \leq G_u$, see Fig. 3)

$$M(G) = \int d\varphi \, R^2(G,\varphi), \tag{21}$$
$$N(G) = \int d\varphi \, R^2(G,\varphi) \left[\mu - R^2(G,\varphi)\right] \; (G \leq G_u)$$
$$R^2(G,\varphi) = \mu - \cos 2\varphi - \left[(\mu - \cos 2\varphi)^2 - 4G\right]^{1/2}.$$

(here, the integral is taken from 0 to π).

C. Explicit limiting expressions for the escape rates

The expressions for the activation energies are simplified near bifurcation points (yet not too close to the bifurcation points, so that the damping of the vibrations with a given g (16) be small compared to their frequency). These bifurcation points are $\mu = \pm 1$.

As the increasing μ goes through the value $\mu = -1$, the maximum of the function $G(X,Y)$ at $X = Y = 0$ becomes a saddle point from which there are split off two maxima of G corresponding to period two attractors (see Fig. 3). With the further increase in μ, for $\mu = 1$ the point $X = Y = 0$ becomes a minimum of $G(X,Y)$ from which there are split off two saddles of G. Escape from the emerging stable states is determined by small-radius orbits $G(X,Y) = G$. For such orbits $M(G)/N(G) \approx 1/\mu$ in (18), and therefore

$$S_1 = S_2 \approx \zeta(\mu+1)^2/2, \; 0 < \mu + 1 \ll 1, \tag{22}$$
$$S_0 \approx \zeta(\mu-1)^2/2, \; 0 < \mu - 1 \ll 1.$$

In the limit of large μ, the activation energy for escape from period two attractors (18) is determined by orbits with $G_u = (\mu-1)^2/4 \leq G \leq G_{1,2} = (\mu+1)^2/4$. These orbits have a shape of narrow arcs on the (X,Y)-plane. It is seen from (20) that, for such orbits $M(G)/N(G) \approx 1/\mu$, and

$$S_1 = S_2 \approx 2\zeta, \; \mu \gg 1. \tag{23}$$

One can also show from (21) that

$$S_0 \approx \zeta\mu, \; \mu \gg 1. \tag{24}$$

In the general case of arbitrary values of the parameter $\mu = \Omega/\zeta$ the activation energies $S_1 = S_2, S_0$ could be found by evaluating the integrals (18), (20), (21) numerically. The results are shown in Fig. 4.

It is seen from Fig. 4 that $S_1 = S_2$ is quadratic in $(\mu+1)$ for small $\mu+1$, and monotonically increases with the increasing μ. For large μ, $S_{1,2}$ saturates at $\approx 2\zeta$. The activation energy S_0 is quadratic in $\mu - 1$ for small $\mu - 1$, and then monotonically increases with the increasing μ. We note that $S_1 > S_0$ for $\mu \lesssim 6.5$. Respectively, for such μ the escape rates from the period two attractors

are *exponentially smaller* than from the state where the vibrations are not excited. As a consequence, the stationary population of the period two attractors $w_1 = w_2$ is exponentially larger than the population w_0 of the steady state:

$$w_1 = w_2 = (W_0/2W_1)w_0,\ w_1/w_0 \propto \exp[(S_1 - S_0)/D]. \tag{25}$$

For larger frequency detuning ($\Omega = \mu\zeta$), the steady state becomes more populated, in agreement with the intuitive physical argument that, as the field is detuned further away from the resonance, it is less likely that the period two vibrations will be excited, for the same field intensity.

V. EFFECTS OF THE SIXTH-ORDER ANHARMONIC TERM

In the experiments [2], because of the structure of the electrostatic field in the trap for an oscillating electron, the 6th order anharmonic term in the Hamiltonian of the electron vibrations could be relatively large (in fact, the 4th order term could be relatively small), while higher-order terms remain much smaller than both the 4th and 6th order terms [17]. The advantage of suppressing the 4th order term is that the amplitude of the period-two vibrations becomes larger. When the 6th order anharmonicity is taken into account, the equation of motion takes the form

$$\frac{d^2q}{dt^2} + 2\Gamma\frac{dq}{dt} + \omega_0^2 q + \gamma q^3 + \lambda q^5 + qF\cos\omega_F t = \xi(t) \tag{26}$$

For the model (26), the equations of motion in the rotating frame are again of the form (4), but now the function $g(q_1, q_2)$ is given by the expression

$$\begin{aligned} g(q_1,q_2) &= \frac{1}{2}\Omega(q_1^2+q_2^2) - \frac{1}{4}(q_1^2+q_2^2)^2\mathrm{sgn}\,\gamma \\ &-\frac{\rho}{6\zeta}(q_1^2+q_2^2)^3 + \frac{1}{2}\zeta(q_2^2-q_1^2), \\ \rho &= \frac{5\lambda F}{9\gamma^2}. \end{aligned} \tag{27}$$

We have neglected the renormalization ($\propto \gamma^2$) of the nonlinearity parameter parameter λ (this renormalization is substantial when γ is not small, in which case the role of the 6th order anharmonicity is insignificant, or the whole approximation of small-amplitude vibrations does not apply). The dimensionless parameter ρ characterizes the "strength" of the 6th order nonlinearity.

For Ω very close to the bifurcation values $\mp(\zeta^2-1)^{1/2}$, escape from the small-amplitude stable state(s) is determined by the motion with small $q_1^2 + q_2^2$. Clearly, this motion is determined by the *lowest-order* anharmonicity, and therefore neither this motion nor the bifurcation values of the parameters are affected by the higher-order nonlinear terms.

In what follows we will investigate the effect of the term $\propto \lambda$ in (26) on the escape rates from the period two attractors, and also from the zero amplitude stable state for $\Omega > (\zeta^2-1)^{1/2}$, far from the bifurcation points. We will analyze the case of weak damping, which is of utmost interest for the experiment [17].

In the neglect of dissipation and fluctuations, the motion of the oscillator (26) in the rotating frame can be described by Eqs. (16), but when the 6th order nonlinearity is taken into account, the effective energy $G(X,Y)$ is of the form

$$\begin{aligned} G(X,Y) &= \frac{1}{2}(\mu-1)X^2 + \frac{1}{2}(\mu+1)Y^2, \\ &-\frac{1}{4}(X^2+Y^2)^2 - \frac{1}{6}\rho(X^2+Y^2)^3, \quad \mu = \frac{\Omega}{\zeta}. \end{aligned} \tag{28}$$

For $\rho > 0$ (i.e., for $\gamma\lambda > 0$ in Eq. (26)), the phase portrait of the conservative motion (16), (28) remains qualitatively the same as that shown in Figs. 2a,b for $\rho = 0$, as does also the topological structure of the surface $G(X,Y)$ in Fig. 3. The centers which correspond to the period two attractors lie on the Y-axis, $X_{1,2} = 0$, they correspond to the maxima of the function $G(X,Y)$, whereas the hyperbolic points which correspond to the unstable period two states lie on the X-axis, $Y_{u1,2} = 0$; they correspond to the saddles of $G(X,Y)$.

The general expression for the activation energy of escape S_n is given by Eq. (18), but now the functions $M(G)$ and $N(G)$ have to be calculated with account taken of the 6th order terms in $G(X,Y)$ (28). In particular,

$$\begin{aligned} N(G) = \iint_{A(G)} dXdY\ &\left[\mu - 2(X^2+Y^2)\right. \\ &\left.- 3\rho(X^2+Y^2)^2\right] \end{aligned} \tag{29}$$

The functions $M(G), N(G)$ can be written as single integrals over the polar angle (cf. Sec. IV B). In the problem of escape from period two attractors, we obtain that, similar to (20),

$$\begin{aligned} M(G) &= \frac{1}{2}\int d\varphi\ \left[R_+^2(G,\varphi) - R_-^2(G,\varphi)\right] \\ N(G) &= \frac{1}{2}\int d\varphi\ \left[Z_+(G,\varphi) - Z_-(G,\varphi)\right], \\ Z_\pm(G,\varphi) &= \mu R_\pm^2(G,\varphi) - R_\pm^4(G,\varphi) - \rho R_\pm^6(G,\varphi) \end{aligned} \tag{30}$$

Here, $R_\pm(G,\varphi)$ are the radii of the trajectories $G(X,Y) = G$ which surround the centers corresponding to the period two attractors. They are given by the real roots of the equation

$$\begin{aligned} &\frac{1}{6}\rho R_\pm^6 + \frac{1}{4}R_\pm^4 - \frac{1}{2}(\mu - \cos 2\varphi)R_\pm^2 + G = 0, \\ &G_{1,2} > G > 0 \text{ for } \mu < 1,\ G_{1,2} > G > G_u \text{ for } \mu > 1. \end{aligned} \tag{31}$$

($G_1 = G_2$ are the values of G in the period two attractors, see Eq. (34)).The limits of the integrals over φ in Eq. (30) are determined from the condition that Eq. (31) has two real roots, $R_- < R_+$.

For the case of escape from the attractor with zero vibration amplitude ($\mathbf{q}$=0) the functions $M(G)$ and $N(G)$ have the form

$$M(G) = \frac{1}{2}\int_0^{2\pi} d\varphi\, R_-^2(G,\varphi), \tag{32}$$
$$N(G) = \frac{1}{2}\int_0^{2\pi} d\varphi\, Z_-(G,\varphi), \quad \mu > 1,\ G_u > G > 0$$

where the function $Z_-(G,\varphi)$ is defined in Eq. (30) (here, G_u is the value of G in the saddle points, see Eq. (34); in the range $0 < G < G_u$ Eq. (31) has only one real root R_-, which determines the function Z_-).

A. Activation energies of escape near bifurcation points

For underdamped systems where the sixth order anharmonicity, the analysis of the activation energies of escape for parameter values close to the bifurcation points is similar to that in Sec. IV C. In the range $0 < \mu+1 \ll 1$ the centers $(X_{1,2}, Y_{1,2})$) which correspond to the period two attractors are close to the saddle point at the origin. In this case $R_\pm \ll 1$ in Eqs. (31), and the terms with $X^2 + Y^2$ in Eq. (29) can be neglected. These terms can be also neglected in the problem of escape from the stable state $X = Y = 0$ for $0 < \mu + 1 \ll 1$, as in this case the hyperbolic points $(X_{u1,2}, Y_{u1,2})$ are close to the stable state. Therefore, in both cases, $M(G)/N(G) \approx 1/\mu$ (cf. Sec. IV C), and it follows from Eq. (18) that the activation energies of escape

$$S_1 = S_2 \approx 2\zeta G_1, \quad 0 < \mu + 1 \ll 1, \tag{33}$$
$$S_0 \approx 2\zeta G_u \quad 0 < \mu - 1 \ll 1$$

Here, we have taken into account that $G(0, 0) = 0$. The explicit expressions for the effective energies $G_1 = G_2$ (the maxima of $G(X,Y)$) and $G_u \equiv G_{u1} = G_{u2}$ (the saddles of $G(X,Y)$) are of the form

$$G_{1,2} = \frac{[-(1+6\rho(\mu+1)] + [1+4\rho(\mu+1)]^{3/2}}{24\rho^2}, \tag{34}$$
$$G_u = \frac{[-(1+6\rho(\mu-1)] + [1+4\rho(\mu-1)]^{3/2}}{24\rho^2}.$$

For weak sixth order nonlinearity $\rho \ll 1$, or just very close to the bifurcation points, so that $\rho(\mu^2-1) \ll 1$ (but the effects of dissipation are still small), the expressions (33), (34) go over into the asymptotic expressions for the activation energies (22), $S_n \propto (\mu^2 - 1)^2$.

1. Strong sixth order nonlinearity

The activation energies (33) as functions of the distance $\mu^2 - 1$ to the bifurcation points display an interesting behavior for large sixth-order nonlinearity, $\rho \gg 1$, in the range where $\mu^2 - 1 \ll 1$ but $\rho(\mu^2 - 1) \gg 1$. It follows from (33), (34) that in this range

$$S_{1,2} \approx \frac{2}{3}\zeta(\mu+1)^{3/2}\rho^{-1/2},\ \mu+1 \ll 1, \quad \rho(\mu+1) \gg 1, \tag{35}$$
$$S_0 \approx \frac{2}{3}\zeta(\mu-1)^{3/2}\rho^{-1/2},\ \mu-1 \ll 1, \quad \rho(\mu-1) \gg 1.$$

The dependence on the distance to the bifurcation point $\mu^2 - 1$ as given by Eq. (35) is described by the power law with the exponent 3/2. In contrast, for weak sixth order nonlinearity (22) the exponent is equal to 2. Eqs. (33), (34) describe both limiting behaviors and the crossover from one of them to the other.

B. Strong sixth order nonlinearity: general case

For strong sixth order nonlinearity, $\rho \gg 1$ and $\rho(\mu^2 - 1) \ll 1$, it is convenient to rescale the dynamical variables,

$$\tilde{X} = \rho^{1/4}X,\ \tilde{Y} = \rho^{1/4}Y,\ \tilde{G} = \rho^{1/2}G \tag{36}$$
$$\tilde{G} = \frac{1}{2}(\mu-1)\tilde{X}^2 + \frac{1}{2}(\mu+1)\tilde{Y}^2 - \frac{1}{6}(\tilde{X}^2+\tilde{Y}^2)^3,$$

and the noise intensity D (10),

$$\tilde{D} = \rho^{1/2}D = \frac{(5\lambda F)^{1/2}}{8\omega_F^3\Gamma^2}\phi(\omega_F/2). \tag{37}$$

The maximal and saddle values of $\tilde{G}$, $\tilde{G}_{1,2}$ and $\tilde{G}_u$, respectively, are given by the simple expressions

$$G_{1,2} = \frac{1}{3}(\mu+1)^{3/2}; \tag{38}$$
$$G_u = 0 \text{ for } -1 < \mu < 1, \quad G_u = \frac{1}{3}(\mu-1)^{3/2} \text{ for } \mu > 1,$$

whereas the expressions for the trajectories in polar coordinates ($\tilde{R} \equiv \rho^{1/4}R, \varphi$) are of the form

$$\tilde{R}_\pm^2(\tilde{G},\varphi) = 2(\mu - \cos 2\varphi)^{1/2}\cos\left[\frac{\theta(\tilde{G},\varphi) \mp \pi}{3}\right], \tag{39}$$
$$\theta(\tilde{G},\varphi) = \arccos\left[3\tilde{G}/(\mu-\cos 2\varphi)^{3/2}\right]$$

With account taken of (39), the expressions (30) for the functions $\tilde{M} = \rho^{-1/2}M, \tilde{N} = \rho^{-1/2}N$ take the form

$$\tilde{M}(\tilde{G}) = \int d\varphi[3(\mu-\cos 2\varphi)]^{1/2}\sin[\theta(\tilde{G},\varphi)/3], \tag{40}$$
$$\tilde{N}(\tilde{G}) = -\int d\varphi(2\mu - 3\cos 2\varphi)\,[3(\mu-\cos 2\varphi)]^{1/2}$$
$$\times \sin[\theta(\tilde{G},\varphi)/3],$$

(the above expressions refer to the trajectories which surround the period two attractors). For the trajectories that surround the zero-amplitude state $\tilde{X} = \tilde{Y} = 0$ we obtain from (32), (39)

$$\tilde{M}(\tilde{G}) = \int_0^{2\pi} d\varphi\, [\mu - \cos 2\varphi]^{1/2} \cos\left[\frac{\theta(\tilde{G},\varphi)+\pi}{3}\right], \quad (41)$$

$$\tilde{N}(\tilde{G}) = -\int_0^{2\pi} d\varphi\, (2\mu - 3\cos 2\varphi)[\mu - \cos 2\varphi]^{1/2} \times \cos\left[\frac{\theta(G,\varphi)+\pi}{3}\right] + 6\pi\tilde{G}.$$

The expression for the escape activation energies $\tilde{S}_n = \rho^{1/2} S_n$ has the same form as Eq. (18) for S_n, with G, M, N in (18) replaced by $\tilde{G}, \tilde{M}, \tilde{N}$, respectively. The escape rate in the variables with tilde has the same form as in Eq. (11),

$$W_n = C \exp(-\tilde{S}_n/\tilde{D}). \quad (42)$$

It follows from Eqs. (27), (36), (42) that, in the limit of large sixth-order nonlinearity, the fourth-order nonlinearity parameter γ drops out of the expressions for the activation energies $\tilde{S}_n$, the reduced noise intensity $\tilde{D}$, and the escape rates. This is in agreement with Eq. (35), which shows explicitly that $S_n \propto \rho^{-1/2}$ for large ρ, and therefore $\tilde{S}_n = \rho^{1/2} S_n$ is independent of γ.

For large frequency detuning, $\mu \gg 1$, it follows from Eqs. (18), (40), (41) that the activation energies $\tilde{S}_{1,2}, \tilde{S}_0$ are of the form

$$\tilde{S}_{1,2} \approx \zeta\mu^{-1/2}, \quad \tilde{S}_0 \approx \zeta\mu^{1/2}, \quad (\mu \gg 1). \quad (43)$$

It is clear from the asymptotic expressions (35), (43) that the activation energy of escape from the period two attractors $\tilde{S}_{1,2}$ is a *nonmonotonic* function of μ, i.e. of the frequency detuning $\omega_F - 2\omega_0$. The decrease of $\tilde{S}_{1,2}$ for large μ can be understood as follows. As we mentioned before, the effective reciprocal "temperature" of the distribution of the system over the energy G, which is given by the $M/N = \tilde{M}/\tilde{N}$, is determined by the ratio of the rates of the drift and diffusion of the oscillator over G (for a Brownian particle this ratio is indeed equal to $1/kT$). The drift coefficient is linear in the characteristic velocity $\dot{X}, \dot{Y}$ of the oscillator in rotating frame, whereas the diffusion is quadratic in this velocity. Therefore when the velocity is large, the ratio M/N becomes small. This happens for large μ. Here, the characteristic time scale for the motion with a given G is set by the reciprocal frequency detuning $1/|\omega_F - 2\omega_0|$, and the ratio $M/N \propto 1/|\omega_F - 2\omega_0| \propto 1/\mu$. On the other hand, the energy interval $G_1 - G_u$ for large μ is increasing with μ sublinearly, if the sixth order nonlinearity is dominating (and linearly, if the fourth order nonlinearity is dominating). Therefore, for large ρ and μ the activation energy decreases with the increasing μ.

The position of the maximum of $\tilde{S}_{1,2}$ and the overall dependence of $\tilde{S}_n$ on μ can be obtained by numerical integration of Eqs. (18), (40), (41). The results are shown in Fig. 5.

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APPENDIX A: WEAK SIXTH ORDER NONLINEARITY

When $\rho\zeta \ll 1$ the zeroth-order in $\rho\zeta$ values of the functions $M(G)$ and $N(G)$ in the Eqs. (30) and (32) are determined by the Eqs.(20) and (21), respectively. In the case of period two attractors to the first order in $\rho\zeta$ the functions M, N have the form

$$M(G) \approx M^{(0)}(G) + M^{(1)}(G), N(G) \approx N^{(0)}(G) + N^{(1)}(G).$$

$$M^{(1)}(G) = -\frac{4\rho\zeta}{3}\int_{\varphi_1}^{\varphi_2} d\varphi\, \frac{1}{f(G,\varphi)}(\mu - \cos 2\varphi) \quad (A1)$$
$$\times[(\mu - \cos 2\varphi)^2 - 3\tilde{G}],$$

$$N^{(1)}(G) = N_1^{(1)}(G) + N_2^{(1)}(G), \quad (A2)$$

$$N_1^{(1)}(G) = -4\rho\zeta \int_{\varphi_1}^{\varphi_2} d\varphi\, f(G,\varphi)[(\mu - \cos 2\varphi)^2 - G],$$

$$N_2^{(1)}(G) = \frac{\rho\zeta}{6}\int_{\varphi_1}^{\varphi_2} d\varphi\, \left[2\left((R_+^{(0)}(G,\varphi))^7 + (R_-^{(0)}(G,\varphi))^7\right)\right.$$
$$\left. -\mu\left((R_+^{(0)}(G,\varphi))^5 + (R_-^{(0)}(G,\varphi))^5\right)\right]/f(G,\varphi)$$

Here $R_\pm^{(0)}$ are the radii of the orbits with a given G evaluated for $\rho = 0$. The angles $\varphi_{1,2}$ in (A1),(A2) are evaluated in the zeroth order in $\rho\zeta$

$$\varphi_1 = \frac{1}{2}\arccos(\mu - 2G^{1/2}), \; \varphi_2 = \pi - \varphi_1, \quad (A3)$$

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Figures

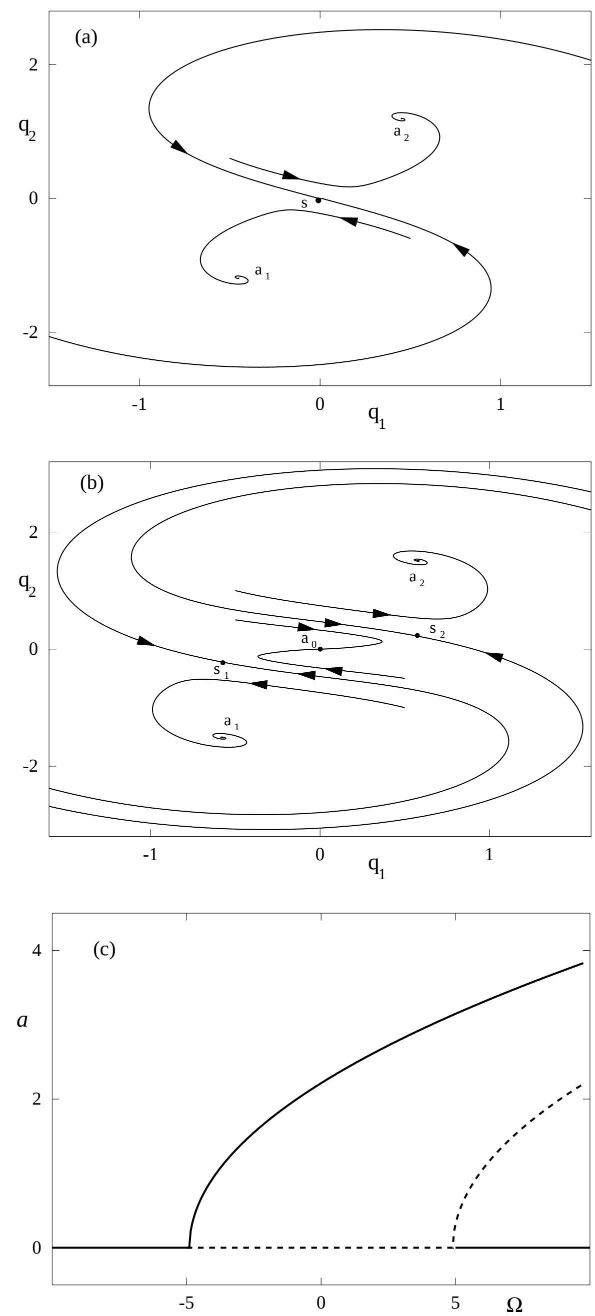


FIG. 1. Trajectories and separatrices of the oscillator in the absence of noise in slow variables q_1, q_2 for (a) $\zeta = 1.5$, $\Omega = 0.5$ where the stable states of the oscillator are period two attractors, and (b) for $\zeta = 1.5$, $\Omega = 1.5$ where the steady state $\mathbf{q} = 0$ is also stable. The positions of the stable states and the saddle points are denoted by the letters a_i and s, respectively. (c) The dependence of the dimensionless amplitude of the stable (solid line) and unstable (dashed line) period two vibrations on Ω for $\zeta = 5.0$.

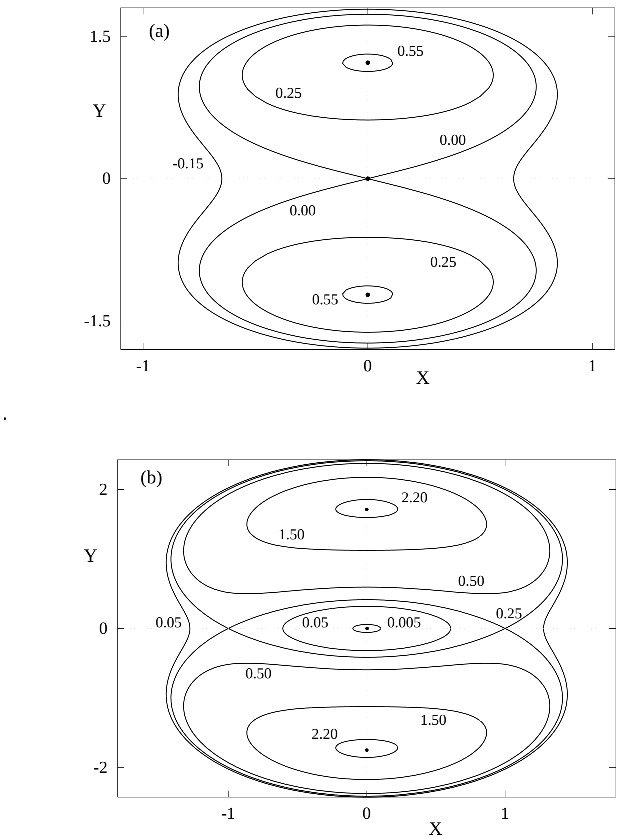
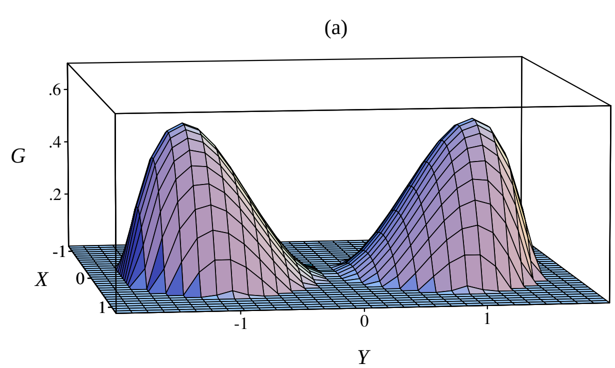


FIG. 2. Trajectories of the conservative motion (16) for (a) $\mu = 0.5$ and (b) $\mu = 2.0$. The values of the Hamiltonian function $G(X, Y)$ are shown near the trajectories. The dots show the positions of the centers and the hyperbolic points.



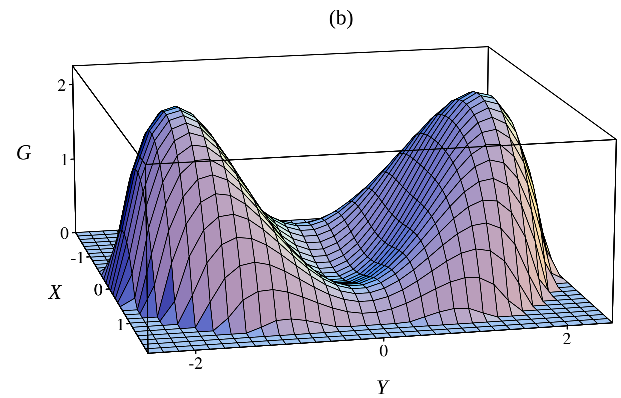


FIG. 3. The Hamiltonian function $G(X, Y)$ (16) (a) for $\mu = 0.5$ where the function G has two maxima (which correspond to period two attractors, with account taken of dissipation) and a saddle point at $X = Y = 0$, and (b) for $\mu = 2.0$ where G has two maxima, a minimum at $X = Y = 0$, and two saddles which correspond to unstable period two vibrations, with account taken of dissipation.

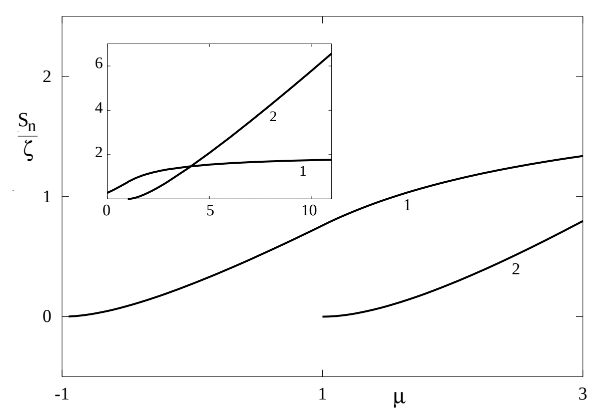


FIG. 4. The dependence of the escape activation energy $S_1 = S_2$ (lines 1) and S_0 (lines 2) on the scaled frequency detuning $\mu = \Omega/\zeta \equiv 2\omega_F[(\omega_F/2) - \omega_0]/F$ in the limit of comparatively large fields or small damping, $\zeta \gg 1$.

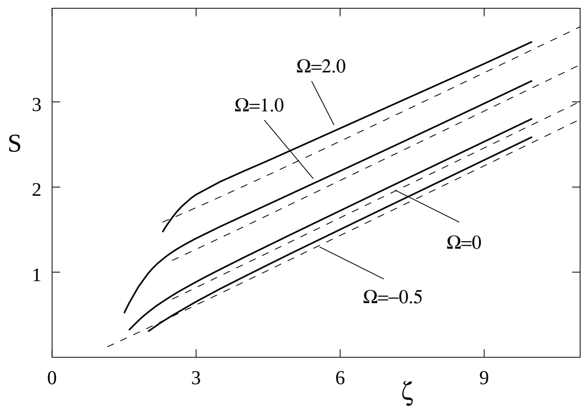


FIG. 5. The dependence of the activation energies of the phase slip transitions between period two attractors on the scaled field amplitude for two different values of the frequency detuning Ω, as obtained by solving the variational problem (11) (solid lines). The dashed lines show the low-damping (large ζ) asymptotes (cf. Fig. 4)

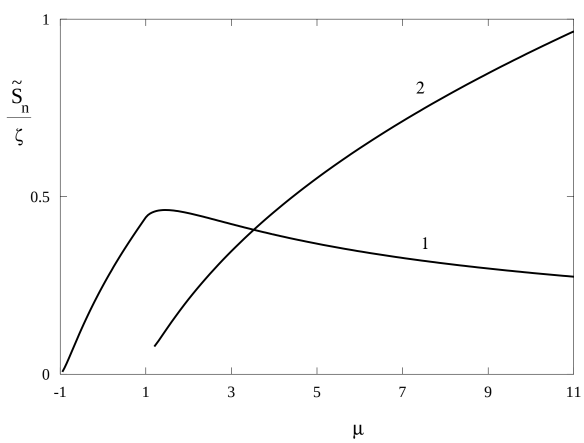


FIG. 6. The dependence of the escape activation energies $\tilde{S}_1 = \tilde{S}_2$ (line 1) and $\tilde{S}_0$ (lines 2) on the scaled frequency detuning $\mu = \Omega/\zeta \equiv 2\omega_F[(\omega_F/2) - \omega_0]/F$ in the limit of comparatively large fields or small damping, $\zeta \gg 1$, for large sixth order nonlinearity, $\rho \gg 1$.